# THE EXTENDED FREUDENTHAL MAGIC SQUARE AND JORDAN ALGEBRAS

ISABEL CUNHA$^{\diamond}$ AND ALBERTO ELDUQUE$^{\star}$

Abstract. The Lie superalgebras in the extended Freudenthal Magic Square in characteristic 3 are shown to be related to some known simple Lie superalgebras, specific to this characteristic, constructed in terms of orthogonal and symplectic triple systems, which are defined in terms of central simple degree three Jordan algebras.

## 1. Introduction

In [Eld06b], some simple finite dimensional Lie superalgebras over fields of characteristic 3, with no counterpart in Kac's classification ([Kac77]), were constructed by means of the so called symplectic and orthogonal triple systems over these fields, most of them related to simple Jordan algebras.

On the other hand, Freudenthal Magic Square, which contains in characteristic 0 the exceptional simple finite dimensional Lie algebras, other than $G_2$, is usually constructed based on two ingredients: a unital composition algebra and a central simple degree 3 Jordan algebra (see [Sch95, Chapter IV]). This construction, due to Tits, does not work in characteristic 3.

A more symmetric construction, based on two unital composition algebras, which play symmetric roles, and their triality Lie algebras, has been given recently by several authors ([AF93], [BS], [LM02]). Among other things, this construction has the advantage of being valid too in characteristic 3. Simpler formulas for triality appear if symmetric composition algebras are used, instead of the more classical unital composition algebras. These algebras permit a simple construction of Freudenthal Magic Square ([Eld04]).

But the characteristic 3 presents an exceptional feature, as only over fields of this characteristic there are nontrivial composition superalgebras, which appear in dimensions 3 and 6. This fact allows to extend Freudenthal Magic Square ([CE]) with the addition of two further rows and columns, filled with (mostly simple) Lie superalgebras.

This paper is devoted to show that the Lie superalgebras (or their derived subalgebras) that appear in this extended Freudenthal Magic Square and

which are constructed in terms of a three dimensional composition super-algebra and a composition algebra are among the simple Lie superalgebras defined in [Eld06b] by means of simple orthogonal triple systems; while those that appear constructed in terms of a six dimensional composition super-algebra and a composition algebra are among those in [Eld06b] defined by means of simple symplectic triple systems. Some reflections will be given too on the simple Lie superalgebras obtained from two nontrivial composition superalgebras.

The paper is structured as follows. Section 2 will be devoted to review the construction of the extended Freudenthal Magic Square in characteristic 3 in [CE]. Then Section 3 will deal with the Lie superalgebras of derivations of the Jordan superalgebras of $3 \times 3$ hermitian matrices over unital compo-sition superalgebras, and their relationship to the first row of the extended Freudenthal Magic Square. Section 4 will will be devoted to orthogonal triple systems, some of the simple Lie superalgebras constructed out of them in [Eld06b], and their connections to the column of the extended Freudenthal Magic Square which correspond to the three dimensional composition su-peralgebras. Section 5 deals with symplectic triple systems and the column attached to the six dimensional composition superalgebras. Finally, in Sec-tion 6, the relationship of the remaining Lie superalgebras in the extended Freudenthal Magic Square to Lie superalgebras in [Eld06b] and [Elda] will be highlighted, as well as their connections to triple systems of a mixed nature: the orthosymplectic triple systems, which will be defined here.

## 2. The extended Freudenthal Magic Square in characteristic 3

A quadratic superform on a $\mathbb{Z}_2$-graded vector space $U = U_{\bar{0}} \oplus U_{\bar{1}}$ over a field $k$ is a pair $q = (q_{\bar{0}}, \mathrm{b})$ where $q_{\bar{0}} : U_{\bar{0}} \to k$ is a quadratic form, and $\mathrm{b} : U \times U \to k$ is a supersymmetric even bilinear form such that $\mathrm{b}|_{U_{\bar{0}} \times U_{\bar{0}}}$ is the polar of $q_{\bar{0}}$ ($\mathrm{b}(x_{\bar{0}}, y_{\bar{0}}) = q_{\bar{0}}(x_{\bar{0}} + y_{\bar{0}}) - q_{\bar{0}}(x_{\bar{0}}) - q_{\bar{0}}(y_{\bar{0}})$ for any $x_{\bar{0}}, y_{\bar{0}} \in U_{\bar{0}}$).

The quadratic superform $q = (q_{\bar{0}}, \mathrm{b})$ is said to be *regular* if $q_{\bar{0}}$ is reg-ular (definition as in [KMRT98, p. xix]) and the restriction of $\mathrm{b}$ to $U_{\bar{1}}$ is nondegenerate.

Then a superalgebra $C = C_{\bar{0}} \oplus C_{\bar{1}}$ over $k$, endowed with a regular qua-dratic superform $q = (q_{\bar{0}}, \mathrm{b})$, called the *norm*, is said to be a *composition superalgebra* (see [EO02]) in case

$$q_{\bar{0}}(x_{\bar{0}} y_{\bar{0}}) = q_{\bar{0}}(x_{\bar{0}}) q_{\bar{0}}(y_{\bar{0}}), \tag{2.1a}$$

$$\mathrm{b}(x_{\bar{0}} y, x_{\bar{0}} z) = q_{\bar{0}}(x_{\bar{0}}) \mathrm{b}(y, z) = \mathrm{b}(y x_{\bar{0}}, z x_{\bar{0}}), \tag{2.1b}$$

$$\mathrm{b}(xy, zt) + (-1)^{|x||y|+|x||z|+|y||z|} \mathrm{b}(zy, xt) = (-1)^{|y||z|} \mathrm{b}(x, z) \mathrm{b}(y, t), \tag{2.1c}$$

for any $x_{\bar{0}}, y_{\bar{0}} \in C_{\bar{0}}$ and homogeneous elements $x, y, z, t \in C$ (where $|x|$ denotes the parity of the homogeneous element $x$).

The unital composition superalgebras are termed *Hurwitz superalgebras*, and a composition superalgebra is said to be *symmetric* in case its bilinear form is associative, that is, $\mathrm{b}(xy, z) = \mathrm{b}(x, yz)$, for any $x, y, z$.

Hurwitz algebras are the well-known algebras that generalize the classical real division algebras of the real and complex numbers, quaternions and octonions. Over any algebraically closed field $k$, there are exactly four of

them: $k$, $k \times k$, $\mathrm{Mat}_2(k)$ and $C(k)$ (the split Cayley algebra), with dimensions 1, 2, 4 and 8.

Only over fields of characteristic 3 there appear nontrivial Hurwitz superalgebras (see [EO02]):

- Let $V$ be a two dimensional vector space over a field $k$, endowed with a nonzero alternating bilinear form $\langle .|.\rangle$. Consider the superspace $B(1,2)$ (see [She97]) with

$$B(1,2)_{\bar{0}} = k1, \qquad \text{and} \qquad B(1,2)_{\bar{1}} = V, \tag{2.2}$$

  endowed with the supercommutative multiplication given by

$$1x = x1 = x \qquad \text{and} \qquad uv = \langle u|v\rangle 1$$

  for any $x \in B(1,2)$ and $u, v \in V$, and with the quadratic superform $q = (q_{\bar{0}}, \mathrm{b})$ given by:

$$q_{\bar{0}}(1) = 1, \quad \mathrm{b}(u,v) = \langle u|v\rangle, \tag{2.3}$$

  for any $u, v \in V$. If the characteristic of $k$ is 3, then $B(1,2)$ is a Hurwitz superalgebra ([EO02, Proposition 2.7]).

- Moreover, with $V$ as before, let $f \mapsto \bar{f}$ be the associated symplectic involution on $\mathrm{End}_k(V)$ (so $\langle f(u)|v\rangle = \langle u|\bar{f}(v)\rangle$ for any $u, v \in V$ and $f \in \mathrm{End}_k(V)$). Consider the superspace $B(4,2)$ (see [She97]) with

$$B(4,2)_{\bar{0}} = \mathrm{End}_k(V), \qquad \text{and} \qquad B(4,2)_{\bar{1}} = V, \tag{2.4}$$

  with multiplication given by the usual one (composition of maps) in $\mathrm{End}_k(V)$, and by

$$v \cdot f = f(v) = \bar{f} \cdot v,$$
$$u \cdot v = \langle .|u\rangle v \ (w \mapsto \langle w|u\rangle v) \ \in \mathrm{End}_k(V),$$

  for any $f \in \mathrm{End}_k(V)$ and $u, v \in V$; and with quadratic superform $q = (q_{\bar{0}}, \mathrm{b})$ such that

$$q_{\bar{0}}(f) = \det(f), \qquad \mathrm{b}(u,v) = \langle u|v\rangle,$$

  for any $f \in \mathrm{End}_k(V)$ and $u, v \in V$. Again, if the characteristic is 3, $B(4,2)$ is a Hurwitz superalgebra ([EO02, Proposition 2.7]).

Given any Hurwitz superalgebra $C$ with norm $q = (q_{\bar{0}}, \mathrm{b})$, its standard involution is given by $x \mapsto \bar{x} = \mathrm{b}(x,1)1 - x$. If $\varphi$ is any automorphism of $C$ with $\varphi^3 = 1$, then a new product can be defined on $C$ by means of

$$x \bullet y = \varphi(\bar{x})\varphi^2(\bar{y}). \tag{2.5}$$

The resulting superalgebra, denoted by $\bar{C}_\varphi$, is called a *Petersson superalgebra* and it turns out to be a symmetric composition superalgebra.

In particular, for $\varphi = 1$, $\bar{C} = \bar{C}_1$ is said to be the *para-Hurwitz superalgebra* attached to $C$.

Over a field $k$ of characteristic 3, consider the Hurwitz superalgebra $B(1,2)$, and take a basis $\{v, w\}$ of $V$ with $\langle v|w\rangle = 1$. Then, for any scalar $\lambda$, the even linear map $\varphi : B(1,2) \to B(1,2)$ such that $\varphi(1) = 1$, $\varphi(v) = v$ and $\varphi(w) = \lambda v + w$, is an order 3 automorphism (or $\varphi = 1$ if $\lambda = 0$). Denote by $S_{1,2}^\lambda$ the symmetric composition superalgebra $\overline{B(1,2)}_\varphi$. Also, denote by $S_{4,2}$ the para-Hurwitz superalgebra $\overline{B(4,2)}$.

Any symmetric composition algebra with a nonzero idempotent is a Petersson superalgebra (see [EP96], [KMRT98, Chapter VIII] and [EO02]) and this is always the case over algebraically closed fields. If $S_r$ ($r = 1$, 2, 4 or 8) denotes the para-Hurwitz algebra attached to the split Hurwitz algebra of dimension $r$ (which is either $k$, $k \times k$, $\mathrm{Mat}_2(k)$ or $C(k)$), and $\tilde{S}_8$ denotes the split Okubo algebra (that is, the pseudo-octonion algebra $P_8(k)$ in [EP96]), which is a particular instance of Petersson algebra constructed from $C(k)$, then (see [EO02, Theorem 4.3]):

**Theorem 2.6.** *Let $k$ be an algebraically closed field of characteristic $3$. Then, up to isomorphism, any symmetric composition superalgebra is one of $S_1$, $S_2$, $S_4$, $S_8$, $\tilde{S}_8$, $S_{1,2}^\lambda$ ($\lambda \in k$), or $S_{4,2}$.*

Given a symmetric composition superalgebra $S$, its *triality Lie superalgebra* $\mathfrak{tri}(S) = \mathfrak{tri}(S)_{\bar{0}} \oplus \mathfrak{tri}(S)_{\bar{1}}$ is defined by:

$$\mathfrak{tri}(S)_{\bar{i}} = \{(d_0, d_1, d_2) \in \mathfrak{osp}(S, q)_{\bar{i}}^3 :$$
$$d_0(x \bullet y) = d_1(x) \bullet y + (-1)^{i|x|} x \bullet d_2(y) \ \forall x, y \in S_{\bar{0}} \cup S_{\bar{1}}\},$$

where $\bar{i} = \bar{0}, \bar{1}$, and $\mathfrak{osp}(S, q)$ denotes the associated orthosymplectic Lie superalgebra. The bracket in $\mathfrak{tri}(S)$ is given componentwise.

Now, given two symmetric composition superalgebras $S$ and $S'$ over a field $k$ of characteristic $\neq 2$, one can form (see [CE, §3]) the Lie superalgebra:

$$\mathfrak{g} = \mathfrak{g}(S, S') = \big(\mathfrak{tri}(S) \oplus \mathfrak{tri}(S')\big) \oplus \big(\oplus_{i=0}^2 \iota_i(S \otimes S')\big),$$

where $\iota_i(S \otimes S')$ is just a copy of $S \otimes S'$ ($i = 0, 1, 2$), with bracket given by:

- $\mathfrak{tri}(S) \oplus \mathfrak{tri}(S')$ is a Lie subsuperalgebra of $\mathfrak{g}$,
- $[(d_0, d_1, d_2), \iota_i(x \otimes x')] = \iota_i\big(d_i(x) \otimes x'\big)$,
- $[(d_0', d_1', d_2'), \iota_i(x \otimes x')] = (-1)^{|d_i'||x|} \iota_i\big(x \otimes d_i'(x')\big)$,
- $[\iota_i(x \otimes x'), \iota_{i+1}(y \otimes y')] = (-1)^{|x'||y|} \iota_{i+2}\big((x \bullet y) \otimes (x' \bullet y')\big)$ (indices modulo 3),
- $[\iota_i(x \otimes x'), \iota_i(y \otimes y')] = (-1)^{|x||x'|+|x||y'|+|y||y'|} \mathrm{b}'(x', y') \theta^i(t_{x,y})$
  $\qquad\qquad + (-1)^{|y||x'|} \mathrm{b}(x, y) \theta'^i(t_{x',y'}')$,

for any $i = 0, 1, 2$ and homogeneous $x, y \in S$, $x', y' \in S'$, $(d_0, d_1, d_2) \in \mathfrak{tri}(S)$, and $(d_0', d_1', d_2') \in \mathfrak{tri}(S')$. Here $\theta$ denotes the natural automorphism $\theta : (d_0, d_1, d_2) \mapsto (d_2, d_0, d_1)$ in $\mathfrak{tri}(S)$, $\theta'$ the analogous automorphism of $\mathfrak{tri}(S')$, and

$$t_{x,y} = \big(\sigma_{x,y}, \tfrac{1}{2}\mathrm{b}(x, y)1 - r_x l_y, \tfrac{1}{2}\mathrm{b}(x, y)1 - l_x r_y\big) \tag{2.7}$$

(with $l_x(y) = x \bullet y$, $r_x(y) = (-1)^{|x||y|} y \bullet x$, $\sigma_{x,y}(z) = (-1)^{|y||z|} \mathrm{b}(x, z) y - (-1)^{|x|(|y|+|z|)} \mathrm{b}(y, z) x$ for homogeneous $x, y, z \in S$), while $\theta'$ and $t_{x',y'}'$ denote the analogous elements for $\mathfrak{tri}(S')$.

This construction is a superization of the algebra construction in [Eld04], which in turn is based on previous constructions (with Hurwitz algebras) in [BS, BS03, LM02, LM04]. It gives a symmetric and simple construction of Freudenthal Magic Square. The advantage of using symmetric composition (super)algebras lies in the simplicity of the formulas needed.

Over an algebraically closed field $k$ of characteristic 3, and because of [Eld06a, Theorem 12.2], it is enough to deal with the Lie superalgebras $\mathfrak{g}(S, S')$, where $S$ and $S'$ are one of $S_1$, $S_2$, $S_4$, $S_8$, $S_{1,2} = S_{1,2}^0$ or $S_{4,2}$ in Theorem 2.6. These are displayed in Table 1, which has been obtained in [CE].

| | $S_1$ | $S_2$ | $S_4$ | $S_8$ | $S_{1,2}$ | $S_{4,2}$ |
|---|---|---|---|---|---|---|
| $S_1$ | $\mathfrak{sl}_2$ | $\mathfrak{pgl}_3$ | $\mathfrak{sp}_6$ | $\mathfrak{f}_4$ | $\mathfrak{psl}_{2,2}$ | $\mathfrak{sp}_6 \oplus (14)$ |
| $S_2$ | | $\mathfrak{pgl}_3 \oplus \mathfrak{pgl}_3$ | $\mathfrak{pgl}_6$ | $\tilde{\mathfrak{e}}_6$ | $\left(\mathfrak{pgl}_3 \oplus \mathfrak{sl}_2\right) \oplus \left(\mathfrak{psl}_3 \otimes (2)\right)$ | $\mathfrak{pgl}_6 \oplus (20)$ |
| $S_4$ | | | $\mathfrak{so}_{12}$ | $\mathfrak{e}_7$ | $\left(\mathfrak{sp}_6 \oplus \mathfrak{sl}_2\right) \oplus \left((13) \otimes (2)\right)$ | $\mathfrak{so}_{12} \oplus spin_{12}$ |
| $S_8$ | | | | $\mathfrak{e}_8$ | $\left(\mathfrak{f}_4 \oplus \mathfrak{sl}_2\right) \oplus \left((25) \otimes (2)\right)$ | $\mathfrak{e}_7 \oplus (56)$ |
| $S_{1,2}$ | | | | | $\mathfrak{so}_7 \oplus 2spin_7$ | $\mathfrak{sp}_8 \oplus (40)$ |
| $S_{4,2}$ | | | | | | $\mathfrak{so}_{13} \oplus spin_{13}$ |

TABLE 1. Freudenthal Magic Supersquare (characteristic 3)

Since the construction of $\mathfrak{g}(S, S')$ is symmetric, only the entries above the diagonal are needed. In Table 1, $\mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ denote the simple exceptional classical Lie algebras, $\tilde{\mathfrak{e}}_6$ denotes a 78 dimensional Lie algebras whose derived Lie algebra is the (77 dimensional) simple Lie algebra $\mathfrak{e}_6$ (the characteristic is 3!). The even and odd parts of the nontrivial superalgebras in the table which have no counterpart in Kac's classification in caracteristic 0 ([Kac77]) are displayed, $spin$ denotes the spin module for the corresponding orthogonal Lie algebra, while $(n)$ denotes a module of dimension $n$. Thus, for example, $\mathfrak{g}(S_4, S_{1,2})$ is a Lie superalgebra whose even part is (isomorphic to) the direct sum of the symplectic Lie algebra $\mathfrak{sp}_6(k)$ and of $\mathfrak{sl}_2(k)$, while its odd part is the tensor product of a 13 dimensional module for $\mathfrak{sp}_6(k)$ and the natural 2 dimensional module for $\mathfrak{sl}_2(k)$.

A precise description of these modules and of the Lie superalgebras as contragredient Lie superalgebras is given in [CE].

The main purpose of this paper is to show the relationships of these superalgebras $\mathfrak{g}(S, S_{1,2})$ and $\mathfrak{g}(S, S_{4,2})$ to some superalgebras constructed in [Eld06b] by means of orthogonal and symplectic triple systems, and strongly related to some simple Jordan algebras.

## 3. SOME JORDAN SUPERALGEBRAS AND THEIR DERIVATIONS

Given any Hurwitz superalgebra $C$ over a field $k$ of characteristic $\neq 2$, with norm $q = (q_{\bar{0}}, \mathrm{b})$ and standard involution $x \mapsto \bar{x}$, the superalgebra $H_3(C, *)$ of $3 \times 3$ hermitian matrices over $C$, where $(a_{ij})^* = (\bar{a}_{ji})$, is a Jordan superalgebra under the symmetrized product

$$x \circ y = \frac{1}{2}\left(xy + (-1)^{|x||y|}yx\right). \tag{3.1}$$

Let us consider the associated para-Hurwitz superalgebra $S = \bar{C}$, with multiplication $a \bullet b = \bar{a}\bar{b}$ for any $a, b \in C$. Then,

$$J = H_3(C, *) = \left\{ \begin{pmatrix} \alpha_0 & \bar{a}_2 & a_1 \\ a_2 & \alpha_1 & \bar{a}_0 \\ \bar{a}_1 & a_0 & \alpha_2 \end{pmatrix} : \alpha_0, \alpha_1, \alpha_2 \in k, \ a_0, a_1, a_2 \in S \right\}$$

$$= \left( \oplus_{i=0}^2 k e_i \right) \oplus \left( \oplus_{i=0}^2 \iota_i(S) \right),$$

where

$$e_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\iota_0(a) = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{a} \\ 0 & a & 0 \end{pmatrix}, \quad \iota_1(a) = 2 \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ \bar{a} & 0 & 0 \end{pmatrix}, \quad \iota_2(a) = 2 \begin{pmatrix} 0 & \bar{a} & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\tag{3.2}$$

for any $a \in S$. Identify $ke_0 \oplus ke_1 \oplus ke_2$ to $k^3$ by means of $\alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2 \simeq (\alpha_0, \alpha_1, \alpha_2)$. Then the supercommutative multiplication (3.1) becomes:

$$\begin{cases} (\alpha_0, \alpha_1, \alpha_2) \circ (\beta_1, \beta_2, \beta_3) = (\alpha_0 \beta_0, \alpha_1 \beta_1, \alpha_2 \beta_2), \\[2mm] (\alpha_0, \alpha_1, \alpha_2) \circ \iota_i(a) = \dfrac{1}{2}(\alpha_{i+1} + \alpha_{i+2})\iota_i(a), \\[2mm] \iota_i(a) \circ \iota_{i+1}(b) = \iota_{i+2}(a \bullet b), \\[2mm] \iota_i(a) \circ \iota_i(b) = 2\mathrm{b}(a, b)\big(e_{i+1} + e_{i+2}\big), \end{cases} \tag{3.3}$$

for any $\alpha_i, \beta_i \in k$, $a, b \in S$, $i = 0, 1, 2$, and where indices are taken modulo 3.

The aim of this section is to show that the Lie superalgebra of derivations of $J$ is naturally isomorphic to the Lie superalgebra $\mathfrak{g}(S_1, S)$.

This is well-known for algebras, as $\mathfrak{g}(S_1, S)$ is isomorphic to the Lie algebra $\mathcal{T}(k, H_3(C, *))$ obtained by means of Tits construction (see [Eld04] and [BS03]), and this latter algebra is, by its own construction, the derivation algebra of $H_3(C, *)$. What will be done in this section is to make explicit this isomorphism $\mathfrak{g}(S_1, S) \cong \mathfrak{der}\, J$ and extend it to superalgebras.

To begin with, (3.3) shows that $J$ is graded over $\mathbb{Z}_2 \times \mathbb{Z}_2$ with:

$$J_{(0,0)} = k^3, \quad J_{(1,0)} = \iota_0(S), \quad J_{(0,1)} = \iota_1(S), \quad J_{(1,1)} = \iota_2(S)$$

and, therefore, $\mathfrak{der}\, J$ is accordingly graded over $\mathbb{Z}_2 \times \mathbb{Z}_2$:

$$(\mathfrak{der}\, J)_{(i,j)} = \{d \in \mathfrak{der}\, J : d\big(J_{(r,s)}\big) \subseteq J_{(i+r, j+s)} \ \forall r, s = 0, 1\}.$$

**Lemma 3.4.** $(\mathfrak{der}\, J)_{(0,0)} = \{d \in \mathfrak{der}\, J : d(e_i) = 0 \ \forall i = 0, 1, 2\}.$

*Proof.* If $d \in (\mathfrak{der}\, J)_{(0,0)}$, then $d(e_i) \in J_{(0,0)}$ for any $i$. But $J_{(0,0)}$ is isomorphic to $k^3$, whose Lie algebra of derivations is trivial. Hence, $d(e_i) = 0$ for any $i$.

Conversely, if $d \in \mathfrak{der}\, J$ and $d(e_i) = 0$ for any $i$, then for any $i = 0, 1, 2$ and any $s \in S = \bar{C}$:

$$d\big(\iota_i(s)\big) = 2d\big(e_{i+1} \circ \iota_i(s)\big) = 2e_{i+1} \circ d\big(\iota_i(s)\big),$$
$$d\big(\iota_i(s)\big) = 2d\big(e_{i+2} \circ \iota_i(s)\big) = 2e_{i+2} \circ d\big(\iota_i(s)\big),$$
$$0 = d\big(e_i \circ \iota_i(s)\big) = e_i \circ d\big(\iota_i(s)\big),$$

so $d\big(\iota_i(s)\big) \in \{x \in J : e_i \circ x = 0,\ e_{i+1} \circ x = \frac{1}{2}x = e_{i+2} \circ x\} = \iota_i(S)$. Hence $d$ preserves the grading: $d \in (\mathfrak{der}\, J)_{(0,0)}$. $\qquad\square$

**Lemma 3.5.** $(\mathfrak{der}\, J)_{(0,0)}$ *is isomorphic to the triality Lie superalgebra* $\mathfrak{tri}(S)$.

*Proof.* For any homogeneous $d \in (\mathfrak{der}\, J)_{(0,0)}$, there are homogeneous linear maps $d_i \in \mathrm{End}_k(S)$, $i = 0, 1, 2$, of the same parity, such that $d\big(\iota_i(s)\big) = \iota_i\big(d_i(s)\big)$ for any $s \in S$ and $i = 0, 1, 2$. Now, for any $a, b \in S$ and $i = 0, 1, 2$:

$$0 = 2\mathrm{b}(a, b)d(e_{i+1} + e_{i+2}) = d\big(\iota_i(a) \circ \iota_i(b)\big)$$
$$= \iota_i\big(d_i(a)\big) \circ \iota_i(b) + (-1)^{|a||d|}\iota_i(a) \circ \iota_i\big(d_i(b)\big)$$
$$= 2\big(\mathrm{b}(d_i(a), b) + (-1)^{|a||d|}\mathrm{b}(a, d_i(b))\big)(e_{i+1} + e_{i+2}),$$

so $d_i$ belongs to the orthosymplectic Lie superalgebra $\mathfrak{osp}(S, \mathrm{b})$. Also,

$$\iota_i\big(d_i(a \bullet b)\big) = d\big(\iota_i(a \bullet b)\big) = d\big(\iota_{i+1}(a) \circ \iota_{i+2}(b)\big)$$
$$= d\big(\iota_{i+1}(a)\big) \circ \iota_{i+2}(b) + (-1)^{|a||d|}\iota_{i+1}(a) \circ d\big(\iota_{i+2}(b)\big)$$
$$= \iota_{i+1}\big(d_{i+1}(a)\big) \circ \iota_{i+2}(b) + (-1)^{|a||d|}\iota_{i+1}(a) \circ \iota_{i+2}\big(d_{i+2}(b)\big)$$
$$= \iota_i\Big(d_{i+1}(a) \bullet b + (-1)^{|a||d|}a \bullet d_{i+2}(b)\Big),$$

which shows that $(d_0, d_1, d_2) \in \mathfrak{tri}(S)$. Now, the linear map

$$\mathfrak{tri}(S) \longrightarrow (\mathfrak{der}\, J)_{(0,0)}$$
$$(d_0, d_1, d_2) \mapsto D_{(d_0, d_1, d_2)},$$

such that

$$\begin{cases} D_{(d_0,d_1,d_2)}(e_i) = 0, \\ D_{(d_0,d_1,d_2)}\big(\iota_i(a)\big) = \iota_i\big(d_i(a)\big) \end{cases} \tag{3.6}$$

for any $i = 0, 1, 2$ and $a \in S$, is clearly an isomorphism. $\qquad\square$

For any $i = 0, 1, 2$ and $a \in S$, consider the following inner derivation of the Jordan superalgebra $J$:

$$D_i(a) = 2\big[L_{\iota_i(a)}, L_{e_{i+1}}\big] \tag{3.7}$$

(indices modulo 3), where $L_x$ denotes the multiplication by $x$ in $J$. Note that the restriction of $L_{e_i}$ to $\iota_{i+1}(S) \oplus \iota_{i+2}(S)$ is half the identity, so the inner derivation $\big[L_{\iota_i(a)}, L_{e_i}\big]$ is trivial on $\iota_{i+1}(S) \oplus \iota_{i+2}(S)$, which generates $J$. Hence

$$\big[L_{\iota_i(a)}, L_{e_i}\big] = 0 \tag{3.8}$$

for any $i = 0, 1, 2$ and $a \in S$. Also, $L_{e_0+e_1+e_2}$ is the identity map, so $\big[L_{\iota_i(a)}, L_{e_0+e_1+e_2}\big] = 0$, and hence

$$D_i(a) = 2\big[L_{\iota_i(a)}, L_{e_{i+1}}\big] = -2\big[L_{\iota_i(a)}, L_{e_{i+2}}\big]. \tag{3.9}$$

A straightforward computation with (3.3) gives

$$D_i(a)(e_i) = 0, \ D_i(a)(e_{i+1}) = \frac{1}{2}\iota_i(a), \ D_i(a)(e_{i+2}) = -\frac{1}{2}\iota_i(a),$$

$$D_i(a)\big(\iota_{i+1}(b)\big) = -\iota_{i+2}(a \bullet b),$$

$$D_i(a)\big(\iota_{i+2}(b)\big) = (-1)^{|a||b|}\iota_{i+1}(b \bullet a),$$

$$D_i(a)\big(\iota_i(b)\big) = 2\mathrm{b}(a,b)(-e_{i+1} + e_{i+2}),$$

(3.10)

for any $i = 0, 1, 2$ and any homogeneous elements $a, b \in S$.

Denote by $D_i(S)$ the linear span of the $D_i(a)$'s, $a \in S$.

**Lemma 3.11.** $D_0(S) = (\mathfrak{der}\, J)_{(1,0)}$, $D_1(S) = (\mathfrak{der}\, J)_{(0,1)}$, and $D_2(S) = (\mathfrak{der}\, J)_{(1,1)}$.

*Proof.* By symmetry, it is enough to prove the first assertion. The containment $D_0(S) \subseteq (\mathfrak{der}\, J)_{(1,0)}$ is clear.

Note that $e_0 + e_1 + e_2$ is the unity element of $J$, and $e_i \circ e_i = e_i$ for any $i$. Thus any $d \in \mathfrak{der}\, J$ satisfies

$$d(e_0 + e_1 + e_2) = 0, \quad \text{and} \quad d(e_i) = 2e_i \circ d(e_i),$$

for any $i = 0, 1, 2$. Hence,

$$d(e_i) \in \{x \in J : e_i \circ x = \frac{1}{2}x\} = \iota_{i+1}(S) \oplus \iota_{i+2}(S).$$

For $d \in (\mathfrak{der}\, J)_{(1,0)}$ one gets

$$d(e_0) \in J_{(1,0)} \cap \big(\iota_1(S) \oplus \iota_2(S)\big) = \iota_0(S) \cap \big(\iota_1(S) \oplus \iota_2(S)\big) = 0,$$

and since $d(e_0 + e_1 + e_2) = 0$, $d(e_1) = -d(e_2)$ follows. Since $d \in (\mathfrak{der}\, J)_{(1,0)}$, there exists an element $a \in S$ such that $d(e_1) = \iota_0(a)$, and then

$$d - D_0(a) \in (\mathfrak{der}\, J)_{(1,0)} \cap \{f \in \mathfrak{der}\, J : f(e_i) = 0 \ \forall i = 0, 1, 2\},$$

and $d = D_0(a)$ by Lemma 3.4.                                                      $\square$

Therefore, the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading of $\mathfrak{der}\, J$ becomes

$$\mathfrak{der}\, J = D_{\mathfrak{tri}(S)} \oplus \big(\oplus_{i=0}^2 D_i(S)\big)$$

(3.12)

On the other hand, $S_1 = k1$, with $1 \bullet 1 = 1$ and $\mathrm{b}(1,1) = 2$, so $\mathfrak{tri}(S_1) = 0$ and for the para-Hurwitz superalgebra $S$:

$$\mathfrak{g}(S_1, S) = \mathfrak{tri}(S) \oplus \big(\oplus_{i=0}^2 \iota_i(S_1 \otimes S)\big)$$

$$= \mathfrak{tri}(S) \oplus \big(\oplus_{i=0}^2 \iota_i(1 \otimes S)\big).$$

**Theorem 3.13.** *Let $S$ be a para-Hurwitz superalgebra over a field of characteristic $\neq 2$ and let $J$ be the Jordan superalgebra of $3 \times 3$ hermitian matrices over the associated Hurwitz superalgebra. Then the linear map:*

$$\Phi : \mathfrak{g}(S_1, S) \longrightarrow \mathfrak{der}\, J,$$

(3.14)

*such that*

$$\Phi\big((d_0, d_1, d_2)\big) = D_{(d_0,d_1,d_2)},$$

$$\Phi\big(\iota_i(1 \otimes a)\big) = D_i(a),$$

*for any $i = 0, 1, 2$, $a \in S$ and $(d_0, d_1, d_2) \in \mathfrak{tri}(S)$, is an isomorphism of Lie superalgebras.*

*Proof.* Equation (3.12) shows that $\Phi$ is an isomorphism of vector spaces. By symmetry it is enough to check that:

(i) $\big[D_{(d_0,d_1,d_2)}, D_{(f_0,f_1,f_2)}\big] = D_{([d_0,f_0],[d_1,f_1],[d_2,f_2])}$,

(ii) $\big[D_{(d_0,d_1,d_2)}, D_i(a)\big] = D_i\big(d_i(a)\big)$,

(iii) $\big[D_0(a), D_1(b)\big] = D_2(a \bullet b)$, and

(iv) $\big[D_0(a), D_0(b)\big] = 2D_{t_{a,b}}$,

for any $(d_0, d_1, d_2), (f_0, f_1, f_2) \in \mathfrak{tri}(S)$ and $a, b \in S$, where $t_{a,b}$ is defined in (2.7).

Both (i) and (ii) are clear from the definitions in (3.6) and (3.7). Now, for any homogeneous elements $a, b \in S$,

$$
\begin{aligned}
\big[D_0(a), D_1(b)\big] &= \big[D_0(a), 2[L_{\iota_1(b)}, L_{e_2}]\big] \\
&= 2\big[L_{D_0(a)(\iota_1(b))}, L_{e_2}\big] + (-1)^{|a||b|} 2\big[L_{\iota_1(b)}, L_{D_0(a)(e_2)}\big] \\
&= -2\big[L_{\iota_2(a\bullet b)}, L_{e_2}\big] - (-1)^{|a||b|}\big[L_{\iota_1(b)}, L_{\iota_0(a)}\big] \\
&= \big[L_{\iota_0(a)}, L_{\iota_1(b)}\big] \quad \text{by (3.8)},
\end{aligned}
$$

which is an inner derivation in $(\mathfrak{der}\,J)_{(1,1)}$. But

$$
\big[L_{\iota_0(a)}, L_{\iota_1(b)}\big](e_0) = \frac{1}{2}\iota_0(a) \circ \iota_1(b) = \frac{1}{2}\iota_2(a \bullet b) = D_2(a \bullet b)(e_0).
$$

which shows, since $(\mathfrak{der}\,J)_{(1,1)} = D_2(S)$ and $D_2(a)(e_0) = \frac{1}{2}\iota_2(a)$ for any $a \in S$, that $\big[L_{\iota_0(a)}, L_{\iota_1(b)}\big] = D_2(a \bullet b)$. Hence (iii) follows.

Finally,

$$
\begin{aligned}
\big[D_0(a), D_0(b)\big] &= \big[D_0(a), 2[L_{\iota_0(b)}, L_{e_1}]\big] \\
&= 2\big[L_{D_0(a)(\iota_0(b))}, L_{e_1}\big] + (-1)^{|a||b|} 2\big[L_{\iota_0(b)}, L_{D_0(a)(e_1)}\big] \\
&= 4\mathrm{b}(a,b)[L_{e_0-e_2}, L_{e_1}] + (-1)^{|a||b|}[L_{\iota_0(b)}, L_{\iota_0(a)}] \\
&= -[L_{\iota_0(a)}, L_{\iota_0(b)}].
\end{aligned}
$$

But for any homogeneous $a, b, x \in S$:

$$
\begin{aligned}
\big[L_{\iota_0(a)}, L_{\iota_0(b)}\big]\big(\iota_1(x)\big) &= \iota_0(a) \circ \big(\iota_0(b) \circ \iota_1(x)\big) - (-1)^{|a||b|}\iota_0(b) \circ \big(\iota_0(a) \circ \iota_1(x)\big) \\
&= \iota_0(a) \circ \iota_2(b \bullet x) - (-1)^{|a||b|}\iota_0(b) \circ \iota_2(a \bullet x) \\
&= (-1)^{|a|(|b|+|x|)}\iota_1\big((b \bullet x) \bullet a\big) - (-1)^{|b||x|}\iota_1\big((a \bullet x) \bullet b\big)
\end{aligned}
$$

(see (3.3)). Now, from (2.1c) and the nondegeneracy and associativity of the bilinear form of $S$:

$$
(b \bullet x) \bullet a + (-1)^{|a||x|+|b||x|+|a||b|}(a \bullet x) \bullet b = (-1)^{|b||x|}\mathrm{b}(a,b)x,
$$

so

$$
\big[L_{\iota_0(a)}, L_{\iota_0(b)}\big]\big(\iota_1(x)\big) = \iota_1\big(-\mathrm{b}(a,b)x + 2r_a l_b(x)\big)
$$

and

$$
[D_0(a), D_0(b)]\big(\iota_1(x)\big) = \iota_1\big(\mathrm{b}(a,b)x - 2r_a l_b(x)\big).
$$

In the same vein, one gets

$$
[D_0(a), D_0(b)]\big(\iota_2(x)\big) = \iota_2\big(\mathrm{b}(a,b)x - 2l_a r_b(x)\big),
$$

so the restriction of $[D_0(a), D_0(b)]$ to $\iota_1(S) \oplus \iota_2(S)$ coincides with the restriction of $2D_{t_{a,b}}$. Since $\iota_1(S) \oplus \iota_2(S)$ generates the superalgebra $J$, (iv) follows. □

The Lie superalgebra $[L_J, L_J]$ is the Lie superalgebra $\mathfrak{inder}\, J$ of inner derivations of $J$. The proof above shows that $(\mathfrak{der}\, J)_{(r,s)} = (\mathfrak{inder}\, J)_{(r,s)}$ for $(r,s) \neq (0,0)$, while

$$(\mathfrak{inder}\, J)_{(0,0)} = \sum_{i=0}^{2} \big[L_{\iota_i(S)}, L_{\iota_i(S)}\big] = D_{\sum_{i=0}^{2} \theta^i(t_{S,S})} = \Phi\big(\sum_{i=0}^{2} \theta^i(t_{S,S})\big)$$

(recall that $\theta\big((d_0, d_1, d_2)\big) = (d_2, d_0, d_1)$ for any $(d_0, d_1, d_2) \in \mathfrak{tri}(S)$).

In characteristic 3, $\mathfrak{tri}(S) = \sum_{i=0}^{2} \theta^i(t_{S,S})$ if $\dim S = 1, 4$ or $8$ ([Eld04]), and the same happens with $\mathfrak{tri}(S_{1,2})$ and $\mathfrak{tri}(S_{4,2})$, because of [CE, Corollaries 2.12 and 2.23], while for $\dim S = 2$, $\mathfrak{tri}(S)$ has dimension 2 and $\sum_{i=0}^{2} \theta^i(t_{S,S}) = t_{S,S}$ has dimension 1 (see [Eld04]). In characteristic $\neq 3$, $\mathfrak{tri}(S) = \sum_{i=0}^{2} \theta^i(t_{S,S})$ always holds. Therefore, the proof above, together with the results in [Eld04] and [CE] gives:

**Corollary 3.15.** *Let $S$ be a para-Hurwitz (super)algebra over a field of characteristic $\neq 2$, and let $J$ be the Jordan (super)algebra of $3 \times 3$ hermitian matrices over the associated Hurwitz (super)algebra. Then $\mathfrak{der}\, J$ is a simple Lie (super)algebra that coincides with $\mathfrak{inder}\, J$ unless the characteristic is 3 and $\dim S = 2$. In this latter case $\mathfrak{inder}\, J$ coincides with $[\mathfrak{der}\, J, \mathfrak{der}\, J]$, which is a codimension 1 simple ideal of $\mathfrak{der}\, J$.*

## 4. Orthogonal triple systems

Orthogonal triple systems were first introduced in [Oku93, Section V]:

**Definition 4.1.** Let $T$ be a vector space over a field $k$ endowed with a nonzero symmetric bilinear form $(.|.) : T \times T \to k$, and a triple product $T \times T \times T \to T$: $(x, y, z) \mapsto [xyz]$. Then $T$ is said to be an *orthogonal triple system* if it satisfies the following identitities:

$$[xxy] = 0 \tag{4.2a}$$

$$[xyy] = (x|y)y - (y|y)x \tag{4.2b}$$

$$[xy[uvw]] = [[xyu]vw] + [u[xyv]w] + [uv[xyw]] \tag{4.2c}$$

$$([xyu]|v) + (u|[xyv]) = 0 \tag{4.2d}$$

for any elements $x, y, u, v, w \in T$.

Equation (4.2c) shows that $\mathfrak{inder}\, T = \operatorname{span}\{[xy.] : x, y \in T\}$ is a subalgebra (actually an ideal) of the Lie algebra $\mathfrak{der}\, T$ of derivations of $T$, whose elements are called *inner derivations*. Because of (4.2b), if $\dim T \geq 2$, then $\mathfrak{der}\, T$ is contained in the orthogonal Lie algebra $\mathfrak{so}\big(T, (.|.)\big)$.

The interesting point about these systems is that they provide a nice construction of Lie superalgebras (see [Eld06b, Theorem 4.5]):

**Theorem 4.3.** *Let $T$ be an orthogonal triple system and let $\big(V, \langle .|. \rangle\big)$ be a two dimensional vector space endowed with a nonzero alternating bilinear*

*form. Let $\mathfrak{s}$ be a Lie subalgebra of $\mathfrak{der}\, J$ containing $\mathfrak{inder}\, T$. Define the superalgebra $\mathfrak{g} = \mathfrak{g}(T, \mathfrak{s}) = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with*

$$\begin{cases} \mathfrak{g}_{\bar{0}} = \mathfrak{sp}(V) \oplus \mathfrak{s} & \textit{(direct sum of ideals)}, \\ \mathfrak{g}_{\bar{1}} = V \otimes T\,, \end{cases}$$

*and superanticommutative multiplication given by:*

- *$\mathfrak{g}_{\bar{0}}$ is a Lie subalgebra of $\mathfrak{g}$;*
- *$\mathfrak{g}_{\bar{0}}$ acts naturally on $\mathfrak{g}_{\bar{1}}$, that is,*

$$[s, v \otimes x] = s(v) \otimes x, \qquad [d, v \otimes x] = v \otimes d(x),$$

  *for any $s \in \mathfrak{sp}(V)$, $d \in \mathfrak{s}$, $v \in V$, and $x \in T$;*
- *for any $u, v \in V$ and $x, y \in T$:*

$$[u \otimes x, v \otimes y] = -(x|y)\gamma_{u,v} + \langle u|v \rangle d_{x,y} \tag{4.4}$$

  *where $\gamma_{u,v} = \langle u|.\rangle v + \langle v|.\rangle u$ and $d_{x,y} = [xy.]$.*

*Then $\mathfrak{g}(T, \mathfrak{s})$ is a Lie superalgebra. Moreover, $\mathfrak{g}(T, \mathfrak{s})$ is simple if and only if $\mathfrak{s}$ coincides with $\mathfrak{inder}\, T$ and $T$ is simple.*

*Conversely, given a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with*

$$\begin{cases} \mathfrak{g}_{\bar{0}} = \mathfrak{sp}(V) \oplus \mathfrak{s} & \textit{(direct sum of ideals)}, \\ \mathfrak{g}_{\bar{1}} = V \otimes T & \textit{(as a module for } \mathfrak{g}_{\bar{0}}), \end{cases}$$

*where $T$ is a module for $\mathfrak{s}$, by $\mathfrak{sp}(V)$-invariance of the Lie bracket, equation (4.4) is satisfied for a symmetric bilinear form $(.|.) : T \times T \to k$ and a skew-symmetric bilinear map $d_{.,.} : T \times T \to \mathfrak{s}$. Then, if $(.|.)$ is not $0$ and a triple product on $T$ is defined by means of $[xyz] = d_{x,y}(z)$, $T$ becomes and orthogonal triple system and the image of $\mathfrak{s}$ in $\mathfrak{gl}(T)$ under the given representation is a subalgebra of $\mathfrak{der}\, T$ containing $\mathfrak{inder}\, T$.*

Given an orthogonal triple system $T$, the Lie superalgebra $\mathfrak{g}(T, \mathfrak{inder}\, T)$ will be denoted simply by $\mathfrak{g}(T)$.

**Remark 4.5.** The bracket (4.4) is not exactly the one that appears in [Eld06b, Equation (4.6)], but this latter multiplied by $-1$. This is just obtained by changing $\langle .|.\rangle$ by its negative in [Eld06b].

The classification of the simple finite dimensional orthogonal triple systems over fields of characteristic $0$ appears in [Eld06b, Theorem 4.7], based on the classification of the simple $(-1, -1)$ balanced Freudenthal Kantor triple systems in [EKO03, Theorem 4.3].

In characteristic $3$, there appears at least a new family of simple orthogonal triple systems (see [Eld06b, Examples 4.20]). Actually, let $C$ be a Hurwitz algebra over a field $k$ of characteristic $3$, let $S$ be the associated para-Hurwitz algebra and let $J = H_3(C, *)$ be the simple Jordan algebra already considered in Section 3. Let $t$ be the natural trace form on $J$ and let $t(x, y) = t(x \circ y)$ for any $x, y \in J$. Let $J_0 = \{x \in J : t(x) = 0\}$ be the set of trace zero elements. Since the characteristic is $3$, $1 = e_0 + e_1 + e_2 \in J_0$. Consider the quotient vector space

$$T_J^o = J_0/k1,$$

with triple product given by:

$$[\hat{x}\hat{y}\hat{z}] = \widehat{[L_x, L_y]}(z), \qquad (4.6)$$

for any $x, y, z \in J_0$, where $\hat{x}$ denotes the class of the element $x \in J$ modulo $k1$, and with nondegenerate symmetric bilinear form given by:

$$(\hat{x}|\hat{y}) = t(x \circ y)$$

for any $x, y \in J_0$. These are well defined maps, and with them $T_J^o$ becomes a simple orthogonal triple system ([Eld06b, Examples 4.20]), and hence $\mathfrak{g}(T_J^o)$ is a simple Lie superalgebra.

The results in [Eld06b, Theorem 4.7(iv) and Theorem 4.23] show that:

- for $\dim C = 1$, $\mathfrak{g}(T_J^o)$ is isomorphic to the simple projective special Lie superalgebra $\mathfrak{psl}_{2,2}(k)$;
- for $\dim C = 2$, $\mathfrak{g}(T_J^o)$ is a simple Lie superalgebra of dimension 24, whose even part is the direct sum of a copy of $\mathfrak{sl}_2(k)$ and of a form of $\mathfrak{psl}_3(k)$;
- for $\dim C = 4$, $\mathfrak{g}(T_J^o)$ is a simple Lie superalgebra of dimension 50, whose even part is the direct sum of a copy of $\mathfrak{sl}_2(k)$ and of a form of the symplectic Lie algebra $\mathfrak{sp}_6(k)$;
- for $\dim C = 8$, $\mathfrak{g}(T_J^o)$ is a simple Lie superalgebra of dimension 105, whose even part is the direct sum of a copy of $\mathfrak{sl}_2(k)$ and of a simple Lie algebra $\mathfrak{f}_4$ of type $F_4$.

With the exception of $\mathfrak{psl}_{2,2}(k)$, none of the above simple Lie superalgebras have counterparts in Kac's classification in characteristic 0 ([Kac77]). For $\dim C = 2$ and $k$ algebraically closed, the simple Lie superalgebra $\mathfrak{g}(T_J^o)$ has recently appeared, in a completely different way, in [BL, 4.2 Theorem], as the Lie superalgebra denoted there by bj.

The definition of the triple product (4.6) shows that the Lie algebra of derivations $\mathfrak{der}\, J$, which leaves invariant the trace form and annihilates $k1$, embeds naturally in $\mathfrak{gl}(T_J^o)$ and, moreover, that $\mathfrak{inder}\, J = \mathrm{span}\{[L_x, L_y] : x, y \in J\}$ maps, under this embedding, onto $\mathfrak{inder}\, T_J^o$.

Although not needed later on, if the dimension of $C$ is 2, 4 or 8, it can be shown that $\mathfrak{der}\, T_J^o$ coincides with $\mathfrak{der}\, J$ as follows. Any $d \in \mathfrak{der}\, T_J^o$ induces an even derivation $\delta$ of $\mathfrak{g}(T_J^o) = \big(\mathfrak{sp}(V) \oplus \mathfrak{inder}\, T_J^o\big) \oplus (V \otimes T_J^o)$, such that $\delta(s) = 0$ for any $s \in \mathfrak{sp}(V)$, $\delta(f) = [d, f]$ for any $f \in \mathfrak{inder}\, T_J^o$, and $\delta(v \otimes x) = v \otimes d(x)$ for any $v \in V$ and $x \in T_J^o$. For $\dim C = 4$ (respectively 8), $\mathfrak{inder}\, T_J^o \simeq \mathfrak{inder}\, J = \mathfrak{der}\, J$ is a form of $\mathfrak{sp}_6(k)$ (respectively, a simple Lie algebra of type $F_4$), whose Killing form is nondegenerate (even in this characteristic). Hence its derivations are all inner, and there exists an inner derivation $\tilde{d} \in \mathfrak{inder}\, T_J^o$ such that the restriction $\delta|_{\mathfrak{inder}\, T_J^o} = \mathrm{ad}\, \tilde{d}$. Hence $\delta - \mathrm{ad}\, \tilde{d}$ is an even derivation of $\mathfrak{g}(T_J^o)$ which is trivial on the even part. By irreducibility of the odd part as a module for the even part, its restriction to the odd part is a scalar, which must be 0. Hence $\delta = \mathrm{ad}\, \tilde{d}$, which shows that $d = \tilde{d}$ on $T_J^o$, and hence $\mathfrak{der}\, T_J^o = \mathfrak{inder}\, T_J^o$. If the dimension of $C$ is 2, then $\mathfrak{inder}\, T_J^o$ is a form of $\mathfrak{psl}_3(k)$, whose derivation algebra is $\mathfrak{pgl}_3(k)$. It follows that $\mathfrak{inder}\, T_J^o \simeq \mathfrak{inder}\, J$ has codimension at most 1 in $\mathfrak{der}\, T_J^o$. Since $\mathfrak{inder}\, J$ is a codimension one ideal in $\mathfrak{der}\, J$ in this case, one gets that $\mathfrak{der}\, T_J^o$ coincides with (the image under the natural embedding of) $\mathfrak{der}\, J$.

Under the conditions above, denote simply by $\mathfrak{g}(J)$ the Lie superalgebra $\mathfrak{g}(T^o_J, \mathfrak{der}\, J) = \left(\mathfrak{sp}(V) \oplus \mathfrak{der}\, J\right) \oplus (V \otimes T^o_J)$. If $\dim C = 1$, 4 or 8, then $\mathfrak{g}(J) = \mathfrak{g}(T^o_J)$ (recall that this is, by definition, the Lie superalgebra $\mathfrak{g}(T^o_J, \mathfrak{inder}\, T^o_J) = \mathfrak{g}(T^o_J, \mathfrak{inder}\, J)$), while for $\dim C = 2$, $\mathfrak{g}(T^o_J) = [\mathfrak{g}(J), \mathfrak{g}(J)]$ is a codimension one ideal in $\mathfrak{g}(J)$.

Now Theorem 3.13 shows that $\mathfrak{der}\, J$ is isomorphic to the Lie algebra $\mathfrak{g}(S_1, S)$, where $S$ is the para-Hurwitz algebra attached to $C$. This isomorphism extends to an isomorphism between the Lie superalgebras $\mathfrak{g}(T^o_J)$ and $\mathfrak{g}(S_{1,2}, S)$, as it will be seen shortly.

The results in [CE, Corollary 2.12] shows that the triality Lie superalgebra of $S_{1,2}$ is

$$\mathfrak{tri}(S_{1,2}) = \{(d, d, d) : d \in \mathfrak{osp}(S_{1,2}, \mathrm{b})\},$$

and hence is isomorphic to the orthosymplectic Lie superalgebra $\mathfrak{osp}(S_{1,2}, \mathrm{b})$, which is spanned by the operators:

$$\sigma_{x,y} : z \mapsto (-1)^{|y||z|}\mathrm{b}(x, z)y - (-1)^{|x|(|y|+|z|)}\mathrm{b}(y, z)x.$$

Also, $S_{1,2} = k1 \oplus V$, $\mathrm{b}(1, 1) = 2\, (= -1)$, $\mathrm{b}|_{V \times V} = \langle .|. \rangle$ is a nonzero alternating form (see (2.3)) and $\mathfrak{sp}(V)$ is spanned by the operators:

$$\gamma_{u,v} : w \mapsto \langle u|w \rangle v + \langle v|w \rangle u.$$

Then (see [CE, (2.15)-(2.17)]):

$$\mathfrak{osp}(S_{1,2}, \mathrm{b})_{\bar{0}} = \sigma_{V,V} \simeq \mathfrak{sp}(V), \quad \mathfrak{osp}(S_{1,2}, \mathrm{b})_{\bar{1}} = \sigma_{1,V} \simeq V,$$

where $\sigma_{V,V}$ is identified to $\mathfrak{sp}(V)$ because $\sigma_{V,V}(1) = 0$ and

$$\sigma_{u,v}|_{V \times V} = -\gamma_{u,v}, \tag{4.7}$$

for any $u, v \in V$, while $\sigma_{1,V}$ is identified to $V$ by means of $\sigma_{1,u} \leftrightarrow u$. Note that ([CE, (2.16)]):

$$\sigma_{1,u}(1) = -u, \qquad \sigma_{1,u}(v) = -\langle u|v \rangle 1, \tag{4.8}$$

for any $u, v \in V$.

Then, for any para-Hurwitz algebra $S$:

$$\mathfrak{g}(S_{1,2}, S) = \left(\mathfrak{tri}(S_{1,2}) \oplus \mathfrak{tri}(S)\right) \oplus \left(\oplus_{i=0}^{2} \iota_i(S_{1,2} \otimes S)\right)$$

$$= \left(\mathfrak{tri}(S_{1,2}) \oplus \mathfrak{tri}(S)\right) \oplus \left(\oplus_{i=0}^{2}\left(\iota_i(k1 \otimes S) \oplus \iota_i(V \otimes S)\right)\right)$$

$$= \left(\mathfrak{tri}(S_{1,2}) \oplus \mathfrak{tri}(S)\right) \oplus \left(\oplus_{i=0}^{2}\iota_i(k1 \otimes S)\right) \oplus \left(\oplus_{i=0}^{2}\iota_i(V \otimes S)\right)$$

$$= \left(\mathfrak{tri}(S_{1,2})_{\bar{0}} \oplus \left(\mathfrak{tri}(S) \oplus \left(\oplus_{i=0}^{2}\iota(k1 \otimes S)\right)\right)\right)$$

$$\oplus \left(\mathfrak{tri}(S_{1,2})_{\bar{1}} \oplus \left(\oplus_{i=0}^{2}\iota_i(V \otimes S)\right)\right)$$

$$\simeq \left(\mathfrak{sp}(V) \oplus \mathfrak{g}(S_1, S)\right) \oplus \left(V \oplus \left(\oplus_{i=0}^{2}\iota_i(V \otimes S)\right)\right),$$

where the identifications $\mathfrak{tri}(S_{1,2})_{\bar{0}} \simeq \sigma_{V,V} \simeq \mathfrak{sp}(V)$ and $\mathfrak{tri}(S_{1,2})_{\bar{1}} \simeq \sigma_{1,V} \simeq V$ are used. Actually,

$$\mathfrak{g}(S_{1,2}, S)_{\bar{0}} \simeq \mathfrak{sp}(V) \oplus \mathfrak{g}(S_1, S) \quad \text{(direct sum of ideals)}$$

$$\mathfrak{g}(S_{1,2}, S)_{\bar{1}} \simeq V \oplus \left(\oplus_{i=0}^{2}\iota_i(V \otimes S)\right).$$

On the other hand,
$$\mathfrak{g}(J) = (\mathfrak{sp}(V) \oplus \mathfrak{der}\, J) \oplus (V \otimes T_J^o),$$
and
$$T_J^o = J_0/k1 = \hat{\iota}(k) \oplus \big(\oplus_{i=0}^2 \hat{\iota}_i(S)\big),$$
where
$$\hat{\iota}(1) = \widehat{e_0 - e_1} = \widehat{e_1 - e_2} = \widehat{e_2 - e_0} \quad (\text{as } e_0 + e_1 + e_2 = 0),$$
$$\hat{\iota}_i(s) = \widehat{\iota_i(s)} \quad \text{for any } i = 0, 1, 2 \text{ and } s \in S$$

(see (3.2)).

**Theorem 4.9.** *Let $S$ be a para-Hurwitz algebra over a field $k$ of character-istic $3$ and let $J$ be the Jordan algebra of $3 \times 3$ hermitian matrices over the associated Hurwitz algebra. Then the linear map*
$$\Psi : \mathfrak{g}(S_{1,2}, S) \longrightarrow \mathfrak{g}(J)$$
*such that*

- $\Psi|_{\mathfrak{sp}(V)}$ *is the identity map* $\mathfrak{sp}(V) \simeq \mathfrak{tri}(S_{1,2})_{\bar{0}} \to \mathfrak{sp}(V)$,
- $\Psi|_{\mathfrak{g}(S_1,S)}$ *is the isomorphism* $\Phi : \mathfrak{g}(S_1, S) \to \mathfrak{der}\, J$ *in Theorem 3.13*,
- $\Psi|_V$ *is the 'identity' map* $V \simeq \mathfrak{tri}(S_{1,2})_{\bar{1}} \to V \otimes \hat{\iota}(k) : u \mapsto u \otimes \hat{\iota}(1)$,
- $\Psi\big(\iota_i(u \otimes s)\big) = u \otimes \hat{\iota}_i(s)$ *for any* $u \in V$, $s \in S$ *and* $i = 0, 1, 2$,

*is a Lie superalgebra isomorphism.*

*Proof.* The proof is obtained by straightforward computations. First, it is clear that $\Psi$ is a bijective linear map, and that it maps isomorphically the even part of $\mathfrak{g}(S_{1,2}, S)$ onto the even part of $\mathfrak{g}(J)$, because $\Phi$ is an isomorphism (Theorem 3.13). In checking that $\Psi([g_{\bar{0}}, g_{\bar{1}}]) = [\Psi(g_{\bar{0}}), \Psi(g_{\bar{1}})]$ for any $g_i \in \mathfrak{g}(S_{1,2}, S)_i$, $i = \bar{0}, \bar{1}$, the only nontrivial instances (up to symmetry) to be checked are the following:

- $[\iota_0(1 \otimes a), \iota_0(u \otimes b)] = \mathrm{b}(a,b) t_{1,u} \simeq \mathrm{b}(a,b) u$ in $\mathfrak{g}(S_{1,2}, S)$ for any $a, b \in S$ and $u \in V \simeq (S_{1,2})_{\bar{1}}$; so $\Psi\big([\iota_0(1 \otimes a), \iota_0(u \otimes b)]\big) = \mathrm{b}(a,b) u \otimes \hat{\iota}(1) = \mathrm{b}(a,b) u \otimes \widehat{e_1 - e_2}$, while

$$[\Psi(\iota_0(1 \otimes a)), \Psi(\iota_0(u \otimes b))] = [\Phi(\iota_0(1 \otimes a)), \Phi(\iota_0(u \otimes b))]$$
$$= [D_0(a), u \otimes \hat{\iota}_0(b)] = u \otimes D_0\widehat{(a)(\iota_0}(b))$$
$$= u \otimes 2\mathrm{b}(a,b)(\widehat{-e_1 + e_2}) = \mathrm{b}(a,b) u \otimes \widehat{e_1 - e_2},$$

because of (3.10).

- $[\iota_0(1 \otimes a), \iota_1(u \otimes b)] = \iota_2(-u \otimes a \bullet b)$ for any $a, b \in S$ and $u \in V$ (as $1 \bullet u = \bar{1}\bar{u} = 1(-u) = -u$), so $\Psi\big([\iota_0(1 \otimes a), \iota_1(u \otimes b)]\big) = -u \otimes \hat{\iota}_2(a \bullet b)$, while

$$[\Psi(\iota_0(1 \otimes a)), \Psi(\iota_1(u \otimes b))] = [D_0(a), u \otimes \hat{\iota}_1(b)]$$
$$= u \otimes D_0\widehat{(a)(\iota_1}(b))$$
$$= -u \otimes \hat{\iota}_2(a \bullet b) \text{ (by (3.10))}.$$

- Similarly, $\Psi\big([\iota_0(1 \otimes a), \iota_2(u \otimes b)]\big) = \Psi\big(\iota_1(u \otimes b \bullet a)\big) = u \otimes \hat{\iota}_1(b \bullet a)$, while $[\Psi(\iota_0(1 \otimes a)), \Psi(\iota_2(u \otimes b))] = [D_0(a), u \otimes \hat{\iota}_2(b)] = u \otimes \hat{\iota}_1(b \bullet a)$ too.

Also, the instances to be checked for two odd elements are the following:

- $\Psi\big([u, \iota_0(v \otimes a)]\big) = -\langle u|v\rangle \Psi(\iota_0(1 \otimes a)) = -\langle u|v\rangle D_0(a)$ for any $u, v \in V$ and $a \in S$, because $u \simeq \sigma_{1,u}$ acts on $v$ as indicated in (4.8). And

$$
\begin{aligned}
[\Psi(u), \Psi(\iota_0(v \otimes a))] &= [u \otimes \hat{\imath}(1), v \otimes \hat{\imath}_0(a)] \\
&= [u \otimes \widehat{e_1 - e_2}, v \otimes \hat{\imath}_0(a)] \\
&= \langle u|v\rangle [L_{e_1 - e_2}, L_{\iota_2(a)}] \\
&= -2\langle u|v\rangle [L_{\iota_0(a)}, L_{e_1}] = -\langle u|v\rangle D_0(a),
\end{aligned}
$$

because of (3.9).

- For any $u, v \in V$ and $a, b \in S$,

$$
\begin{aligned}
\Psi\big([\iota_0(u \otimes a), \iota_0(v \otimes b)]\big) &= \Psi\big(\mathrm{b}(a,b)\sigma_{u,v} + \langle u|v\rangle t_{a,b}\big) \\
&= -\mathrm{b}(a,b)\gamma_{u,v} + \langle u|v\rangle D_{t_{a,b}},
\end{aligned}
$$

because of (4.7), while

$$
\begin{aligned}
[\Psi(\iota_0(u \otimes a)), \Psi(\iota_0(v \otimes b))] &= [u \otimes \hat{\imath}_0(a), v \otimes \hat{\imath}_0(b)] \\
&= -t(\iota_0(a) \circ \iota_0(b))\gamma_{u,v} + \langle u|v\rangle [L_{\iota_0(a)}, L_{\iota_0(b)}] \\
&= -\mathrm{b}(a,b)\gamma_{u,v} - \langle u|v\rangle [D_0(a), D_0(b)] \\
&= -\mathrm{b}(a,b)\gamma_{u,v} + \langle u|v\rangle D_{t_{a,b}},
\end{aligned}
$$

because of the formulas in the proof of Theorem 3.13.

- Finally, for $u, v \in V$ and $a, b \in S$,

$$
\begin{aligned}
\Psi\big([\iota_0(u \otimes a), \iota_1(v \otimes b)]\big) &= \Psi\big(\iota_2(\langle u|v\rangle 1 \otimes a \bullet b)\big) \\
&= \langle u|v\rangle \Phi(\iota_2(1 \otimes a \bullet b)) \\
&= \langle u|v\rangle D_2(a \bullet b),
\end{aligned}
$$

while

$$
\begin{aligned}
[\Psi(\iota_0(u \otimes a)), \Psi(\iota_1(v \otimes b))] &= [u \otimes \hat{\imath}_0(a), v \otimes \hat{\imath}_1(b)] \\
&= \langle u|v\rangle [L_{\iota_0(a)}, L_{\iota_1(b)}] \\
&= \langle u|v\rangle [D_0(a), D_1(b)] \\
&= \langle u|v\rangle D_2(a \bullet b),
\end{aligned}
$$

again by the arguments in the proof of Theorem 3.13. $\qquad\square$

This result immediately shows that the Lie superalgebras $\mathfrak{g}(S_{1,2}, S_r)$ in [CE] are, essentially, the new simple Lie superalgebras in [Eld06b, Theorem 4.23]. More specifically:

**Corollary 4.10.** *Let $S_r$ $(r = 1, 2, 4, 8)$ denote the unique para-Hurwitz superalgebra of dimension $r$ over an algebraically closed field $k$ of characteristic 3. Then:*

(i) $\mathfrak{g}(S_{1,2}, S_1)$ *is isomorphic to the classical Lie superalgebra $\mathfrak{psl}_{2,2}(k)$.*

(ii) $[\mathfrak{g}(S_{1,2}, S_2), \mathfrak{g}(S_{1,2}, S)]$ *is isomorphic to the simple Lie superalgebra in [Eld06b, Theorem 4.23(i)], obtained as $\mathfrak{g}(T_J^o)$ for $J = H_3(k \times k, *)$.*

(iii) $\mathfrak{g}(S_{1,2}, S_4)$ *is isomorphic to the simple Lie superalgebra in [Eld06b, Theorem 4.23(iii)], obtained as $\mathfrak{g}(T_J^o)$ for $J = H_3(\mathrm{Mat}_2(k), *)$.*

(iv) $\mathfrak{g}(S_{1,2}, S_8)$ *is isomorphic to the simple Lie superalgebra in [Eld06b, Theorem 4.23(iv)], obtained as $\mathfrak{g}(T_J^o)$ for $J = H_3(C(k), *)$.*

## 5. SYMPLECTIC TRIPLE SYSTEMS

Symplectic triple systems appeared for the first time in [YA75]:

**Definition 5.1.** Let $T$ be a vector space over a field $k$ endowed with a nonzero alternating bilinear form $(.|.) : T \times T \to k$, and a triple product $T \times T \times T \to T$: $(x, y, z) \mapsto [xyz]$. Then $T$ is said to be a *symplectic triple system* if it satisfies the following identities:

$$[xyz] = [yxz] \tag{5.2a}$$

$$[xyz] - [xzy] = (x|z)y - (x|y)z + 2(y|z)x \tag{5.2b}$$

$$[xy[uvw]] = [[xyu]vw] + [u[xyv]w] + [uv[xyw]] \tag{5.2c}$$

$$([xyu]|v) + (u|[xyv]) = 0 \tag{5.2d}$$

for any elements $x, y, z, u, v, w \in T$.

These systems are strongly related to Freudenthal triple systems and to Faulkner ternary algebras (see [Eld06b, Theorems 2.16 and 2.18] and references therein).

As for orthogonal triple systems, for any $x, y$ in a symplectic triple system, the linear map $[xy.]$ is a derivation, and the span $\mathfrak{inder}\, T$ of these derivations is an ideal of $\mathfrak{der}\, T$, whose elements are called *inner derivations*. Then (see [Eld06b, Theorem 2.9]):

**Theorem 5.3.** *Let $T$ be a symplectic triple system and let $\left(V, \langle.|.\rangle\right)$ be a two dimensional vector space endowed with a nonzero alternating bilinear form. Let $\mathfrak{s}$ be a subalgebra of $\mathfrak{der}\, T$ containing $\mathfrak{inder}\, T$. Define the $\mathbb{Z}_2$-graded algebra $\mathfrak{g} = \mathfrak{g}(T, \mathfrak{s}) = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with*

$$\begin{cases} \mathfrak{g}_{\bar{0}} = \mathfrak{sp}(V) \oplus \mathfrak{s} & \text{(direct sum of ideals)}, \\ \mathfrak{g}_{\bar{1}} = V \otimes T, \end{cases}$$

*and anticommutative multiplication given by:*

- *$\mathfrak{g}_{\bar{0}}$ is a Lie subalgebra of $\mathfrak{g}$,*
- *$\mathfrak{g}_{\bar{0}}$ acts naturally on $\mathfrak{g}_{\bar{1}}$; that is*

$$[s, v \otimes x] = s(v) \otimes x, \qquad [d, v \otimes x] = v \otimes d(x),$$

  *for any $s \in \mathfrak{sp}(V)$, $d \in \mathfrak{s}$, $v \in V$, and $x \in T$.*
- *For any $u, v \in V$ and $x, y \in T$:*

$$[u \otimes x, v \otimes y] = (x|y)\gamma_{u,v} + \langle u|v\rangle d_{x,y} \tag{5.4}$$

  *where, as before, $\gamma_{u,v} = \langle u|.\rangle v + \langle v|.\rangle u$ and $d_{x,y} = [xy.]$.*

*Then $\mathfrak{g}(T, \mathfrak{s})$ is a Lie algebra. Moreover, $\mathfrak{g}(T, \mathfrak{s})$ is simple if and only if $\mathfrak{s}$ coincides with $\mathfrak{inder}\, T$ and $T$ is simple.*

*Conversely, given a $\mathbb{Z}_2$-graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with*

$$\begin{cases} \mathfrak{g}_{\bar{0}} = \mathfrak{sp}(V) \oplus \mathfrak{s} & \text{(direct sum of ideals)}, \\ \mathfrak{g}_{\bar{1}} = V \otimes T & \text{(as a module for } \mathfrak{g}_{\bar{0}}\text{)}, \end{cases}$$

*where $T$ is a module for $\mathfrak{s}$, by $\mathfrak{sp}(V)$-invariance of the Lie bracket, equation (5.4) is satisfied for an alternating bilinear form $(.|.) : T \times T \to k$ and a symmetric bilinear map $d_{.,.} : T \times T \to \mathfrak{s}$. Then, if $(.|.)$ is not $0$ and a triple product on $T$ is defined by means of $[xyz] = d_{x,y}(z)$, $T$ becomes a symplectic*

*triple system, and the image of $\mathfrak{s}$ in $\mathfrak{gl}(T)$ under the given representation is a subalgebra of $\mathfrak{der}\,T$ containing $\mathfrak{inder}\,T$.*

The classification of the simple finite dimensional symplectic triple systems in characteristic 3 appears in [Eld06b, Theorem 2.32], based on the classification of Freudenthal triple systems in this characteristic by Brown [Bro84]. The symplectic triple systems which most interest us are the following. Let $C$ be a Hurwitz algebra over a field $k$ (characteristic $\neq 2$), and let $J$ be the Jordan algebra of $3 \times 3$ hermitian matrices over $C$, as in Section 2, with the usual trace $t$. Consider the 'cross product' on $J$ define by

$$x \times y = 2x \circ y + t(x)y + t(y)x - s(x,y)1,$$

where $s(x,y) = t(x)t(y) - t(x \circ y)$. Take the vector space

$$T_J^s = \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} : \alpha, \beta \in k,\ a, b \in J \right\}, \tag{5.5}$$

endowed with the alternating bilinear form and triple product such that, for $x_i = \begin{pmatrix} \alpha_i & a_i \\ b_i & \beta_i \end{pmatrix} \in T_J^s$, $i = 1, 2, 3$:

$$(x_1 | x_2) = \alpha_1 \beta_2 - \alpha_2 \beta_1 - t(a_1, b_2) + t(b_1, a_2),$$

$$[x_1 x_2 x_3] = \begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix} \quad \text{with}$$

$$\gamma = \Big( -3(\alpha_1 \beta_2 + \alpha_2 \beta_1) + t(a_1, b_2) + t(a_2, b_1) \Big)\alpha_3$$

$$+ 2\Big( \alpha_1 t(b_2, a_3) + \alpha_2 t(b_1, a_3) - t(a_1 \times a_2, a_3) \Big)$$

$$c = \Big( -(\alpha_1 \beta_2 + \alpha_2 \beta_1) + t(a_1, b_2) + t(a_2, b_1) \Big)a_3$$

$$+ 2\Big( \big(t(b_2, a_3) - \beta_2 \alpha_3\big)a_1 + \big(t(b_1, a_3) - \beta_1 \alpha_3\big)a_2 \Big)$$

$$+ 2\Big( \alpha_1 (b_2 \times b_3) + \alpha_2 (b_1 \times b_3) + \alpha_3 (b_1 \times b_2) \Big)$$

$$- 2\Big( (a_1 \times a_2) \times b_3 + (a_1 \times a_3) \times b_2 + (a_2 \times a_3) \times b_1 \Big)$$

$$\delta = -\gamma^\sigma,\ d = -c^\sigma, \text{ where } \sigma = (\alpha\beta)(ab) \text{ (that is, } \gamma^\sigma \text{ and } c^\sigma$$

are obtained from $\gamma$ and $c$ by interchanging $\alpha$ and $\beta$ and

also $a$ and $b$ throughout).   $\square$

In characteristic 3 it turns out that any symplectic triple system is an anti-Lie triple system, and hence the next result ([Eld06b, Theorem 3.1]) holds:

**Theorem 5.6.** *Let $T$ be a symplectic triple system over a field $k$ of characteristic 3 and let $\mathfrak{s}$ be a subalgebra of $\mathfrak{der}\,T$ containing $\mathfrak{inder}\,T$. Define the superalgebra $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(T, \mathfrak{s}) = \tilde{\mathfrak{g}}_{\bar{0}} \oplus \tilde{\mathfrak{g}}_{\bar{1}}$, with:*

$$\tilde{\mathfrak{g}}_{\bar{0}} = \mathfrak{s}, \qquad \tilde{\mathfrak{g}}_{\bar{1}} = T,$$

*and superanticommutative multiplication given by:*

- *$\tilde{\mathfrak{g}}_{\bar{0}}$ is a Lie subalgebra of $\tilde{\mathfrak{g}}$;*
- *$\tilde{\mathfrak{g}}_{\bar{0}}$ acts naturally on $\tilde{\mathfrak{g}}_{\bar{1}}$: $[d, x] = d(x)$ for any $d \in \mathfrak{s}$ and $x \in T$;*
- *$[x, y] = d_{x,y} = [xy.]$, for any $x, y \in T$.*

*Then $\tilde{\mathfrak{g}}(T,\mathfrak{s})$ is a Lie superalgebra. Moreover, $\tilde{\mathfrak{g}}(T,\mathfrak{s})$ is simple if and only if so is $T$ and $\mathfrak{s} = \mathfrak{inder}\, T$.*

That is, in Theorem 5.3 one may "delete" $\mathfrak{sp}(V)$ and $V$ and then the $\mathbb{Z}_2$-graded Lie algebra there becomes a Lie superalgebra!

In case $\mathfrak{s} = \mathfrak{inder}\, T$, the Lie superalgebra $\tilde{\mathfrak{g}}(T,\mathfrak{s})$ will just be denoted by $\tilde{\mathfrak{g}}(T)$.

For the symplectic triple systems $T_j^s$ in (5.5), the results in [Eld06b, Theorem 3.2] show that:

- If $\dim C = 1$, then $\tilde{\mathfrak{g}}(T_j^s)$ is a simple Lie superalgebra of dimension 35, whose even part is a form of the symplectic Lie algebra $\mathfrak{sp}_6(k)$ and whose odd part is an irreducible module of dimension 14 for the even part.
- If $\dim C = 2$, then $\tilde{\mathfrak{g}}(T_j^s)$ is a simple Lie superalgebra of dimension 54, whose even part is a form of the projective special Lie algebra $\mathfrak{psl}_6(k)$ and whose odd part is an irreducible module of dimension 20 for the even part.
- If $\dim C = 4$, then $\tilde{\mathfrak{g}}(T_j^s)$ is a simple Lie superalgebra of dimension 98, whose even part is a form of the orthogonal Lie algebra $\mathfrak{so}_{12}(k)$ and whose odd part is an irreducible module of dimension 32 (the spin module) for the even part.
- If $\dim C = 8$, then $\tilde{\mathfrak{g}}(T_j^s)$ is a simple Lie superalgebra of dimension 189, whose even part is a simple exceptional Lie algebra of type $E_7$ and whose odd part is an irreducible module of dimension 56 for the even part.

None of the above simple Lie superalgebras have counterparts in Kac's classification in characteristic 0.

Since any derivation of $\mathfrak{sp}_6(k)$, $\mathfrak{so}_{12}(k)$ or the simple Lie algebras of type $E_7$ is inner, it follows, as in Section 4, that $\mathfrak{inder}\, T_j^s = \mathfrak{der}\, T_j^s$ for $\dim C = 1$, 4 or 8. For $\dim C = 2$, $\mathfrak{der}\, \mathfrak{psl}_6(k) = \mathfrak{pgl}_6(k)$. Extending scalars to an algebraic closure, the symplectic triple system $T_j^s$ is then isomorphic to the one denoted by $\mathcal{T}_{\mathfrak{e}_6,\mathfrak{a}_5}$ in [Eldb, §4], which satisfies that $\mathfrak{a}_5 = \mathfrak{pgl}_6(k)$ is contained in its Lie algebra of derivations. It follows that $\mathfrak{der}\, T_j^s$ is a form of $\mathfrak{pgl}_6(k)$ and $\mathfrak{inder}\, T_j^s = [\mathfrak{der}\, T_j^s, \mathfrak{der}\, T_j^s]$ is a codimension one ideal in $\mathfrak{der}\, T_j^s$ in this case and, therefore, $\tilde{\mathfrak{g}}(T_j^s) = [\tilde{\mathfrak{g}}(T_j^s, \mathfrak{der}\, T_j^s), \tilde{\mathfrak{g}}(T_j^s, \mathfrak{der}\, T_j^s)]$ is a codimension one ideal of $\tilde{\mathfrak{g}}(T_j^s, \mathfrak{der}\, T_j^s)$.

Assume now that the ground field $k$ is algebraically closed of characteristic 3. Then the Lie superalgebras in the extended Freudenthal Magic Square (Table 1) have been constructed in [CE] by means of copies of a two dimensional vector space $V$, endowed with a nonzero alternating bilinear form, and copies of the three dimensional simple Lie algebra $\mathfrak{sp}(V)$, following the approach in [Eldb] used for the Lie algebras in Freudenthal Magic Square.

In particular, if $S_r$ denotes the unique para-Hurwitz algebra of dimension $r$ over $k$ ($r = 1, 2, 4$ or $8$), the construction of the Lie algebra $\mathfrak{g}(S_r, S_8)$ uses four copies of $V$ to deal with $S_8$, say $V_1, V_2, V_3, V_4$. Then (see [Eldb, §4], but with the indices $1, 2, 3, 4$ for $S_8$ in all cases):

$$\mathfrak{g}(S_r, S_8) = \big(\mathfrak{g}(S_r, S_4) \oplus \mathfrak{sp}(V_4)\big) \oplus \Big(\big(\oplus_{\sigma \in \mathcal{S}_{S_r,S_8} \setminus \mathcal{S}_{S_r,S_4}} \tilde{V}(\sigma \setminus \{4\})\big) \otimes V_4\Big), \quad (5.7)$$

where $\mathcal{S}_{S_8,S_8} = \mathcal{S}_{\mathfrak{e}_8}$, $\mathcal{S}_{S_4,S_8} = \mathcal{S}_{\mathfrak{e}_7}$, $\mathcal{S}_{S_2,S_8} = \mathcal{S}_{\mathfrak{e}_6}$, and $\mathcal{S}_{S_1,S_8} = \mathcal{S}_{\mathfrak{f}_4}$ in [Eldb], and $\tilde{V} = V$ unless $r = 2$.

According to Theorem 5.3, this shows that $\oplus_{\sigma \in \mathcal{S}_{S_r,S_8} \setminus \mathcal{S}_{S_r,S_4}} \tilde{V}(\sigma \setminus \{4\})$ is a symplectic triple system and $\mathfrak{g}(S_r, S_4)$ is contained in its Lie algebra of derivations. These are precisely, up to isomorphism, the simple symplectic triple systems $T_J^s$ considered so far (see [Eldb, §4]).

However, a case by case inspection in [CE, §5] shows that for any $r = 1, 2, 4, 8$,

$$\mathcal{S}_{S_r,S_{4,2}} = \mathcal{S}_{S_r,S_4} \cup \{\sigma \setminus \{4\} : \sigma \in \mathcal{S}_{S_r,S_8} \setminus \mathcal{S}_{S_r,S_4}\},$$

and that, if $\mathfrak{sp}(V_4)$ and $V_4$ are "deleted" in (5.7), one obtains precisely the Lie superalgebra $\mathfrak{g}(S_r, S_{4,2})$. These are simple with the exception of $r = 2$, where $[\mathfrak{g}(S_r, S_{4,2}), \mathfrak{g}(S_r, S_{4,2})]$ is a codimension one simple ideal (see [CE]). Therefore, $\mathfrak{g}(S_r, S_{4,2})$ is, up to isomorphism, the Lie superalgebra $\tilde{\mathfrak{g}}(T_J^s, \mathfrak{der}\, T_J^s)$:

**Theorem 5.8.** *Let $k$ be an algebraically closed field of characteristic* 3*, let $S$ be a para-Hurwitz algebra over $k$, let $J$ be the Jordan algebra of $3 \times 3$ hermitian matrices over the associated Hurwitz algebra, and let $T_J^s$ be the symplectic triple system attached to $J$ as in* (5.5)*. Then the Lie superalgebras $\mathfrak{g}(S, S_{4,2})$ and $\tilde{\mathfrak{g}}(T_J^s, \mathfrak{der}\, T_J^s)$ are isomorphic.*

Therefore, the Lie superalgebras $\mathfrak{g}(S, S_{4,2})$ in the extended Freudenthal Magic Square, for a para-Hurwitz algebra $S$, are essentially the new simple Lie superalgebras in [Eld06b, Theorem 3.2]. More specifically:

**Corollary 5.9.** *Let $S_r$ ($r = 1, 2, 4, 8$) denote the unique para-Hurwitz algebra of dimension $r$ over an algebraically closed field $k$ of characteristic* 3*. Then:*

- *$\mathfrak{g}(S_1, S_{4,2})$ is isomorphic to the simple Lie superalgebra in [Eld06b, Theorem 3.2(ii)], obtained as $\mathfrak{g}(T_J^s)$, for $J = H_3(k, *)$.*
- *$[\mathfrak{g}(S_2, S_{4,2}), \mathfrak{g}(S_2, S_{4,2})]$ is isomorphic to the simple Lie superalgebra in [Eld06b, Theorem 3.2(iii)], obtained as $\mathfrak{g}(T_J^s)$, for $J = H_3(k \times k, *)$.*
- *$\mathfrak{g}(S_4, S_{4,2})$ is isomorphic to the simple Lie superalgebra in [Eld06b, Theorem 3.2(iv)], obtained as $\mathfrak{g}(T_J^s)$, for $J = H_3(\mathrm{Mat}_2(k), *)$.*
- *$\mathfrak{g}(S_8, S_{4,2})$ is isomorphic to the simple Lie superalgebra in [Eld06b, Theorem 3.2(v)], obtained as $\mathfrak{g}(T_J^s)$, for $J = H_3(C(k), *)$.*

## 6. Final remarks

In the last sections, the Lie superalgebras $\mathfrak{g}(S_r, S_{1,2})$ and $\mathfrak{g}(S_r, S_{4,2})$ ($r = 1, 2, 4, 8$) in the extended Freudenthal Magic Square have been shown to be related to Lie superalgebras previously constructed in terms of orthogonal and symplectic triple systems in [Eld06b].

Let us have a look in this last section to the remaining Lie superalgebras in the extended square.

The result in [CE, Corollary 5.20] shows that the even part of $\mathfrak{g}(S_{1,2}, S_{1,2})$ is isomorphic to the orthogonal Lie algebra $\mathfrak{so}_7(k)$, while its odd part is the direct sum of two copies of the spin module for $\mathfrak{so}_7(k)$, and therefore, over any algebraically closed field of characteristic 3, $\mathfrak{g}(S_{1,2}, S_{1,2})$ is isomorphic to the simple Lie superalgebra in [Eld06b, Theorem 4.23(ii)], attached to a simple *null orthogonal triple system*.

Actually, the description of $\mathfrak{g} = \mathfrak{g}(S_{1,2}, S_{1,2})$ in [CE, §5.7] shows that

$$\mathfrak{g}_{\bar{0}} = \big(\mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2) \oplus \mathfrak{sl}_2(k)\big) \oplus \big(V_1 \otimes V_2 \otimes \mathfrak{sl}_2(k)\big),$$
$$\mathfrak{g}_{\bar{1}} = \big(V_1 \oplus V_2\big) \otimes \mathfrak{gl}_2(k),$$

and from here it is easy to directly check that $\mathfrak{g}_{\bar{0}}$ is (isomorphic to) the orthogonal Lie algebra $\mathfrak{so}(\mathcal{V}, \mathrm{b})$, where $\mathcal{V} = \big(V_1 \otimes V_2\big) \oplus \mathfrak{sl}_2(k)$ and b is the bilinear form (of maximal Witt index) on $\mathcal{V}$ such that:

$$\mathrm{b}(V_1 \otimes V_2, \mathfrak{sl}_2(k)) = 0,$$
$$\mathrm{b}(u_1 \otimes u_2, v_1 \otimes v_2) = \langle u_1 | v_1 \rangle \langle u_2 | v_2 \rangle,$$
$$\mathrm{b}(p, q) = \det(p + q) - \det(p) - \det(q),$$

for any $u_i, v_i \in V$ $(i = 1, 2)$ and $p, q \in \mathfrak{sl}_2(k)$, and that the Clifford algebra $Clif(\mathcal{V}, -\mathrm{b})$ is isomorphic to $\mathrm{End}_k(\mathfrak{g}_{\bar{1}})$, so that $\mathfrak{g}_{\bar{1}}$ is isomorphic, as a module for $\mathfrak{g}_{\bar{0}} \cong \mathfrak{so}(\mathcal{V}, -\mathrm{b}) \subseteq Clif(\mathcal{V}, -\mathrm{b})_{\bar{0}}$, to the direct sum of two copies of the spin module (see [Cun06] for details).

As for $\mathfrak{g}(S_{4,2}, S_{4,2})$, the results in [CE, Proposition 5.10 and Corollary 5.11] show that its even part is isomorphic to the orthogonal Lie algebra $\mathfrak{so}_{13}(k)$, while its odd part is the spin module for the even part, and hence that $\mathfrak{g}(S_{4,2}, S_{4,2})$ is the simple Lie superalgebra in [Elda, Theorem 3.1(ii)] for $l = 6$.

Only the simple Lie superalgebra $\mathfrak{g}(S_{1,2}, S_{4,2})$ has not previously appeared in the literature. Its even part is isomorphic to the symplectic Lie algebra $\mathfrak{sp}_8(k)$, while its odd part is the irreducible module of dimension 40 which appears as a subquotient of the third exterior power of the natural module for $\mathfrak{sp}_8(k)$ (see [CE, §5.5]).

The results on orthogonal triple systems in Section 4, where $\mathfrak{g}(S_{1,2}, S)$ ($S$ a para-Hurwitz algebra) is shown to be isomorphic to the Lie superalgebra $\mathfrak{g}(T)$ attached to the orthogonal triple system defined on $J_0/k1$ for the Jordan algebra of $3 \times 3$ hermitian matrices on the Hurwitz algebra associated to $S$, suggest the next definition in order to extend this description of $\mathfrak{g}(S_{1,2}, S)$ to the situation in which $S$ is a para-Hurwitz superalgebra.

**Definition 6.1.** Let $T = T_{\bar{0}} \oplus T_{\bar{1}}$ be a vector superspace over a field $k$ of characteristic $\neq 2$ endowed with an even nonzero supersymmetric bilinear form $(.|.) : T \times T \to k$ (that is, $(T_{\bar{0}} | T_{\bar{1}}) = 0$, $(.|.)$ is symmetric on $T_{\bar{0}}$ and alternating on $T_{\bar{1}}$) and a triple product $T \times T \times T \to T$: $(x, y, z) \mapsto [xyz]$ $([x_i y_j z_k] \in T_{i+j+k}$ for any $x_i \in T_i$, $y_j \in T_j$, $z \in T_k$, $i, j, k = \bar{0}$ or $\bar{1})$. Then $T$ is said to be an *orthosymplectic triple system* if it satisfies the following identities:

$$[xyz] + (-1)^{|x||y|}[yxz] = 0 \tag{6.2a}$$

$$[xyz] + (-1)^{|y||z|}[xzy] = (x|y)z + (-1)^{|y||z|}(x|z)y - 2(y|z)x \tag{6.2b}$$

$$[xy[uvw]] = [[xyu]vw] + (-1)^{(|x|+|y|)|u|}[u[xyv]w]$$
$$+ (-1)^{(|x|+|y|)(|u|+|v|)}[uv[xyw]] \tag{6.2c}$$

$$([xyu]|v) + (-1)^{(|x|+|y|)|u|}(u|[xyv]) = 0 \tag{6.2d}$$

for any homogeneous elements $x, y, u, v, w \in T$.

The same arguments as in [Eld06b, Theorem 2.9], with suitable parity signs here and there give:

**Theorem 6.3.** *Let $T$ be an orthosymplectic triple system and let $(V, \langle .|.\rangle)$ be a two dimensional vector space endowed with a nonzero alternating bilinear form. Let $\mathfrak{s}$ be a Lie subalgebra of $\mathfrak{der}\, J$ containing $\mathfrak{inder}\, T$. Define the $\mathbb{Z}_2$-graded superalgebra $\mathfrak{g} = \mathfrak{g}(T, \mathfrak{s}) = \mathfrak{g}(0) \oplus \mathfrak{g}(1)$ with*

$$\begin{cases} \mathfrak{g}(0) = \mathfrak{sp}(V) \oplus \mathfrak{s} & (so \ \mathfrak{g}(0)_{\bar{0}} = \mathfrak{sp}(V) \oplus \mathfrak{s}_{\bar{0}}, \ \mathfrak{g}(0)_{\bar{1}} = \mathfrak{s}_{\bar{1}}), \\ \mathfrak{g}(1) = V \otimes T & (with \ \mathfrak{g}(1)_{\bar{0}} = V \otimes T_{\bar{1}}, \ \mathfrak{g}(1)_{\bar{1}} = V \otimes T_{\bar{0}}, \ V \ is \ odd!), \end{cases}$$

*and superanticommutative multiplication given by:*

- $\mathfrak{g}(0)$ *is a subalgebra of $\mathfrak{g}$;*
- $\mathfrak{g}(0)$ *acts naturally on $\mathfrak{g}_{\bar{1}}$:*

$$[s, v \otimes x] = s(v) \otimes x, \qquad [d, v \otimes x] = (-1)^{|d|} v \otimes d(x),$$

  *for any $s \in \mathfrak{sp}(V)$, $d \in \mathfrak{s}$, $v \in V$, and $x \in T_{\bar{0}} \cup T_{\bar{1}}$;*
- *for any $u, v \in V$ and homogeneous $x, y \in T$:*

$$[u \otimes x, v \otimes y] = (-1)^{|x|}\big(-(x|y)\gamma_{u,v} + \langle u|v\rangle d_{x,y}\big) \tag{6.4}$$

  *where $\gamma_{u,v} = \langle u|.\rangle v + \langle v|.\rangle u$ and $d_{x,y} = [xy.]$.*

*Then $\mathfrak{g}(T, \mathfrak{s})$ is a $\mathbb{Z}_2$-graded Lie superalgebra. Moreover, $\mathfrak{g}(T, \mathfrak{s})$ is simple if and only if $\mathfrak{s}$ coincides with $\mathfrak{inder}\, T$ and $T$ is simple.*

*Conversely, given a $\mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{g} = \mathfrak{g}(0) \oplus \mathfrak{g}(1)$ with*

$$\begin{cases} \mathfrak{g}(0) = \mathfrak{sp}(V) \oplus \mathfrak{s}, \\ \mathfrak{g}(1) = V \otimes T, \end{cases}$$

*where $T$ is a module for the superalgebra $\mathfrak{s}$ and $V$ is considered as an odd vector space, by $\mathfrak{sp}(V)$-invariance of the bracket, equation (6.4) is satisfied for an even supersymmetric bilinear form $(.|.) : T \times T \to k$ and a super-skewsymmetric bilinear map $d_{.,.} : T \times T \to \mathfrak{s}$. Then, if $(.|.)$ is not $0$ and a triple product on $T$ is defined by means of $[xyz] = d_{x,y}(z)$, $T$ becomes and orthosymplectic triple system and the image of $\mathfrak{s}$ in $\mathfrak{gl}(T)$ under the given representation is a subalgebra of $\mathfrak{der}\, T$ containing $\mathfrak{inder}\, T$.*

Now, given a para-Hurwitz superalgebra $S$, let $J$ be the Jordan superalgebra of $3 \times 3$ hermitian matrices over the associated Hurwitz algebras as in Section 2. Consider, as in Section 4, the quotient vector superspace

$$T_J^{os} = J_0/k1 \tag{6.5}$$

with even supersymmetric bilinear form induced by the trace:

$$(\hat{x}|\hat{y}) = t(x \circ y)$$

($\hat{x} = x + k1 \in T_J^{os}$ for $x \in J_0$), and triple product given by

$$[\hat{x}\hat{y}\hat{z}] = [\widehat{L_x, L_y}](z) = x \circ \widehat{(y \circ z)} - (-1)^{|x||y|} y \circ \widehat{(x \circ z)}.$$

Following the ideas in Section 4, consider the $\mathbb{Z}_2$-graded anticommutative superalgebra

$$\mathfrak{g} = \mathfrak{g}(J) = \mathfrak{g}(0) \oplus \mathfrak{g}(1),$$

with $\mathfrak{g}(0) = \mathfrak{sp}(V) \oplus \mathfrak{der}\, J$, and $\mathfrak{g}(1) = V \otimes T_J^{os}$.

The proof of Theorem 4.9, with parity signs put all over, works here to give that $\mathfrak{g}(S_{1,2}, S)$ is isomorphic to $\mathfrak{g}(J)$. Therefore, $\mathfrak{g}(J)$ is a Lie superalgebra!, because so is $\mathfrak{g}(S_{1,2}, S)$, and by Theorem 6.3, $T_J^{os}$ is an orthosymplectic triple system, and $\mathfrak{g}(J) = \mathfrak{g}(T_J^{os}, \mathfrak{der}\, J)$.

Therefore, there appears the following description of the Lie superalgebras $\mathfrak{g}(S_{1,2}, S)$ for any Hurwitz superalgebra $S$, which includes a description of the Lie superalgebras $\mathfrak{g}(S_{1,2}, S_{1,2})$ and $\mathfrak{g}(S_{1,2}, S_{4,2})$ in the extended Freudenthal Magic Square 1.

**Theorem 6.6.** *Let $S$ be a para-Hurwitz superalgebra over a field $k$ of characteristic 3, and let $J$ be the Jordan superalgebra of $3 \times 3$ hermitian matrices over the associated Hurwitz superalgebra. Then the Lie superalgebra $\mathfrak{g}(S_{1,2}, S)$ is isomorphic to the $\mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{g}(J) = \mathfrak{g}(T_J^{os}, \mathfrak{der}\, J)$ associated to the simple orthosymplectic triple system $T_J^{os}$ in (6.5).*

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Departamento de Matemática, Universidade da Beira Interior, 6200 Covilhá, Portugal
  *E-mail address*: icunha@mat.ubi.pt

Departamento de Matemáticas, Universidad de Zaragoza, 50009 Zaragoza, Spain
  *E-mail address*: elduque@unizar.es